Uitwerking Final Exam
Kwantumfysica 1 28 juni 2007

Problem 1

1a) $\lambda = \frac{h}{p}$ $p = \sqrt{2 m E_{k_0}}$
kinetic energy

$$\lambda = \frac{h}{\sqrt{2m E_{k_0}}} = 1.7 \cdot 10^{-11} \text{ m}$$

using $h = 6.626 \cdot 10^{-34}$ Js, $m = 9.1 \cdot 10^{-31}$ kg, $E_{k_0} = 5000$ eV

$1 \text{ eV} = 1.602 \cdot 10^{-19}$ J

1b) Like 1a), but the kinetic energy is reduced due to the higher potential energy U for electrons between the plates.

$U = -eV_\varphi$ (a negative V_φ increases U, so here $-e = -|e| = -1.602 \cdot 10^{-19}$ C is used)

Now $E_{kin} = E_{k_0} - U = E_{k_0} + eV_\varphi$

$\Rightarrow$ as in 1a), $\lambda = \frac{h}{p} = \frac{h}{\sqrt{2m(E_{k_0} + eV_\varphi)}}$

1c) For $V_\varphi = 0$, there must be an interference maximum since the setup is symmetric around the line $y = 0$.

Immediately after the screen, the electron state that propagates to the detector is the superposition state

$$|\Psi_{as}\rangle = \frac{1}{\sqrt{2}}\left(|\Psi_L\rangle + e^{i\theta}|\Psi_R\rangle\right) = \frac{1}{\sqrt{2}}\left(|\Psi_L\rangle + |\Psi_R\rangle\right),$$

where $\theta = 0$ (zero initial phase difference)

and the probability amplitudes are $\frac{1}{\sqrt{2}}$ because of the symmetry of the system.

After the phase controller, at the detector entrance, this state evolved into

$$|\Psi_d\rangle = \frac{1}{\sqrt{2}}\left(|\Psi_L\rangle + e^{i\varphi}|\Psi_R\rangle\right)$$

(where a global phase without physical meaning is neglected)

The count rate r is now

$$r \propto \langle \Psi_d | \Psi_d \rangle = \frac{1}{2}\Big(\langle \Psi_L|\Psi_L\rangle + \langle \Psi_R|\Psi_R\rangle + e^{i\varphi}\langle\Psi_L|\Psi_R\rangle + e^{-i\varphi}\langle \Psi_R|\Psi_L\rangle\Big),$$

where $\langle \Psi_L|\Psi_L\rangle = \langle \Psi_R|\Psi_R\rangle = \langle \Psi_L|\Psi_R\rangle = \langle \Psi_R|\Psi_L\rangle = 1$.

Note that one should simply use this inner product $\langle \Psi_d | \Psi_d \rangle$ since this is equivalent to

$$\int_{\substack{\text{detector}\\ \text{entrance}}} \Psi_d^*(x,y,z)\,\Psi_d(x,y,z)\,dx\,dy\,dz ,$$

which is the inner product describing the probability density of electrons coming into the detector, in x-, y-, z-representation.

So,

$$r \propto \langle \Psi_d | \Psi_d \rangle$$

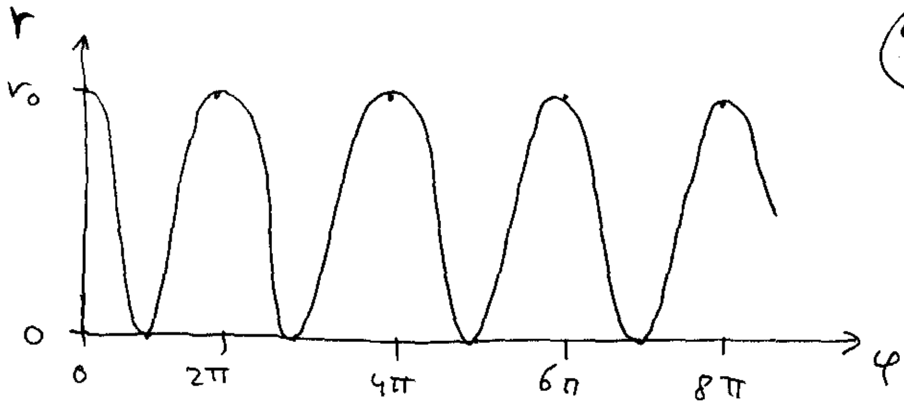
$$= \frac{1}{2}\left(1 + 1 + e^{i\varphi} + e^{-i\varphi} \right)$$

$$= 1 + \cos\varphi$$

For $\varphi = 0$, there is an interference maximum with $r = r_0$, so

$$r = \frac{1}{2} r_0 \left(1 + \cos\varphi \right)$$

(this gives $r = r_0$ for $\varphi = 0$)



1d) The probability <u>amplitude</u> of the right trajectory is reduced by a factor $A_T = \sqrt{T} = \sqrt{0.64} = 0.8$, because of scattering, when electrons in this trajectory pass the phase controller. Now the state at the detector entrance is

$$|\Psi_d\rangle = \frac{1}{\sqrt{2}}\left(|\Psi_L\rangle + e^{i\varphi} A_T |\Psi_R\rangle \right)$$

Thus, the count rate r is now

$$r = \frac{1}{2} r_0 \langle \Psi_d | \Psi_d \rangle$$

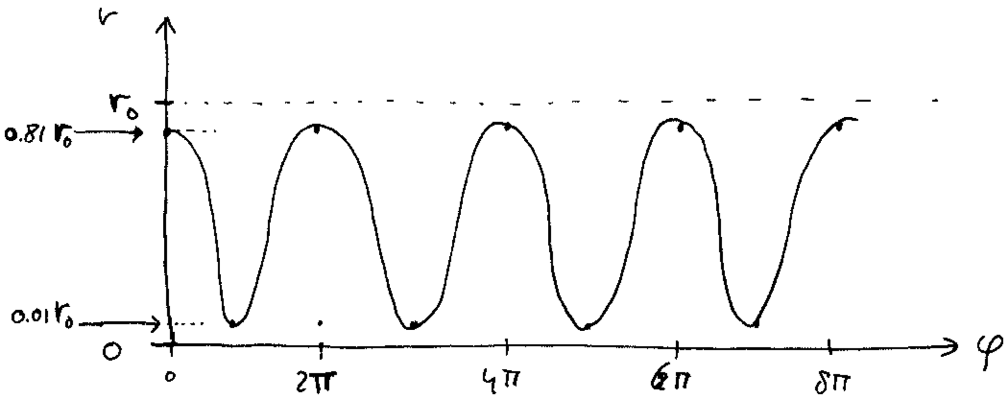
$$= \frac{1}{2} r_0 \; \frac{1}{2}\Big(\langle \Psi_L | \Psi_L \rangle + A_T^* A_T \langle \Psi_R | \Psi_R \rangle + e^{i\varphi} A_T \langle \Psi_L | \Psi_R \rangle + e^{-i\varphi} A_T^* \langle \Psi_R | \Psi_L \rangle \Big)$$

where we must use $A_T = A_T^*$, a real number.

This gives

$$r = \frac{1}{4} r_0 \left(1 + A_T^2 + 2A_T \cos\varphi \right)$$

$$= \frac{1}{4} r_0 \left(1.64 + 1.6 \cos\varphi \right)$$



1e) This is equivalent to question 1d) with $A_T = \sqrt{T} = 0$. $\Rightarrow$

Now $r = \frac{1}{4} r_0 = 250$ counts/sec

(One factor $\frac{1}{2}$ reduction comes from closing the right slit, a second factor $\frac{1}{2}$ reduction comes from smearing out all interference maximums and minimums to one level)

1f) For an individual electron coming to the detector, its contribution to the count rate is

$$r_i \propto 1 + \cos(\varphi + \Delta\varphi_i)$$ (See question 1c)

where i is an index to label each electron, and $-\Delta\varphi < \Delta\varphi_i < \Delta\varphi$, with $\varphi + \Delta\varphi_i$ the real total phase for that electron. Note that $\Delta\varphi_i$ is random and different for each electron, so we must average over all contributions r_i to get the total count rate r.

$$r = \frac{1}{2} r_0 \frac{1}{2\Delta\varphi} \int_{-\Delta\varphi}^{\Delta\varphi} \left(1 + \cos(\varphi + \xi)\right) d\xi$$

ξ representing $\Delta\varphi_i$

$$= \frac{1}{2} r_0 \frac{1}{2\Delta\varphi} \int_{\varphi-\Delta\varphi}^{\varphi+\Delta\varphi} 1 + \cos(q)\, dq \qquad q = \varphi + \xi$$

$$= \frac{1}{2} r_0 \frac{1}{2\Delta\varphi} \Big[q + \sin(q) \Big]_{\varphi-\Delta\varphi}^{\varphi+\Delta\varphi}$$

$$= \frac{1}{2} r_0 \frac{1}{2\Delta\varphi}(\varphi+\Delta\varphi-\varphi+\Delta\varphi) + \frac{1}{2} r_0 \frac{\sin(\varphi+\Delta\varphi)-\sin(\varphi-\Delta\varphi)}{2\Delta\varphi} \Rightarrow$$

$$\boxed{r = \frac{1}{2} r_0 + \frac{1}{2} r_0 \cos(\varphi) \frac{\sin(\Delta\varphi)}{\Delta\varphi}}$$

with $\Delta\varphi = 0.05\,\varphi$, this gives

$$r = \underbrace{\frac{1}{2} r_0 + \frac{1}{2} r_0 \cos(\varphi)}_{\text{Interference pattern of question 1c)}} \underbrace{\frac{\sin(0.05\,\varphi)}{0.05\,\varphi}}_{\text{slow modulation of } \cos(\varphi) \text{ amplitude by sinc function}}$$

For $\varphi = 2\pi$ this gives

$$r = \frac{1}{2} r_0 + \frac{1}{2} r_0 \cdot 1 \cdot \frac{\sin(0.1\,\pi)}{0.1\,\pi} = 0.9918\, r_0$$

Which is ≈ 992 counts per second

1g) Using the result of 1f)

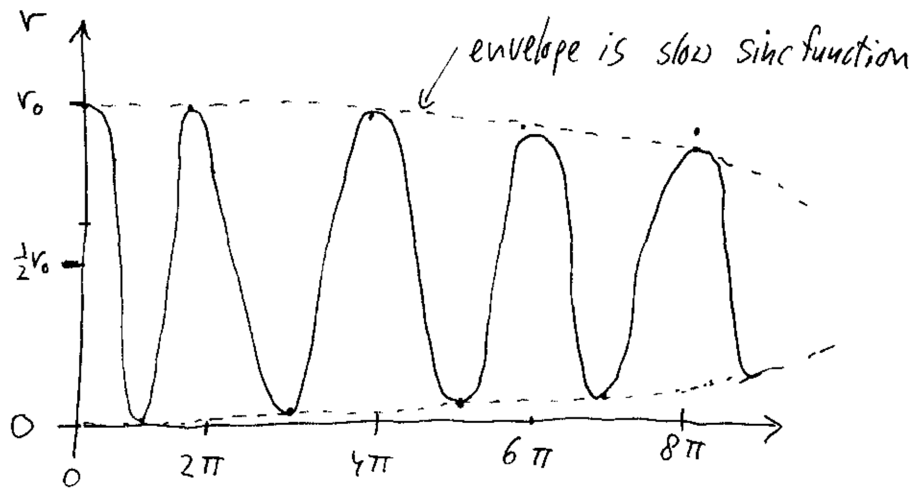
φ	$\frac{r}{r_0} = \frac{1}{2} + \frac{1}{2}\cos\varphi \frac{\sin(0.05\varphi)}{0.05\varphi}$
0	1
2π	0.9918
4π	0.9677
6π	0.9220
8π	0.8784

(Qualitatively, one could argue that the amplitude of the interference reduces to 0 when $\Delta\varphi$ becomes equal to π (that is, $2\Delta\varphi = 2\pi$)

That is $0.05\,\varphi = \pi \Rightarrow \varphi = 20\pi \Rightarrow$

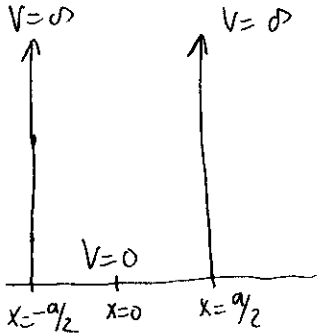
for 8π, the interference amplitude is reduced by about a factor $\approx \frac{20-8}{20} = \frac{12}{20} \approx 0.6$)

With all interference gone, the results goes towards $r = \frac{r_0}{2}$



Problem 2

2a) Solve the time-independent Schrödinger equation to find the energy eigenvalues and -states.



the Hamiltonian reads

$$\begin{cases} \hat{H} = \frac{\hat{p}^2}{2m} + \infty \,, & x < -\frac{a}{2} \\ \hat{H} = \frac{\hat{p}^2}{2m} \,, & -\frac{a}{2} < x < +\frac{a}{2} \\ \hat{H} = \frac{\hat{p}^2}{2m} + \infty \,, & x > +\frac{a}{2} \end{cases}$$

$$H \varphi_n(x) = E_n \varphi_n(x)$$

Look for solutions in the interval $-\frac{a}{2} < x < \frac{a}{2}$ only, since $V = +\infty$ outside this interval.

This also brings the boundary conditions that

$$\varphi_n(x) = 0 \quad \text{for} \quad x = -\frac{a}{2} \quad \text{and} \quad x = +\frac{a}{2}.$$

In this interval, the Schrödinger eq. then reads (with $\hat{p} = -i\hbar \frac{\partial}{\partial x}$)

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \varphi_n(x)}{\partial x^2} = E_n \varphi_n(x) \quad \Rightarrow \text{rewrite as}$$

$$\frac{\partial^2 \varphi_n(x)}{\partial x^2} + k_n^2 \varphi_n = 0 \,, \quad \text{with } k_n = \frac{2mE_n}{\hbar^2}$$

Solutions must be a superposition of harmonic functions with one specific value of k_n (since it has a one-to-one relation with the eigen value E_n), because the differential equation has solutions that have its second derivative equal to the function itself, times a constant.

The most general solution is then

$$\varphi_n(x) = A e^{ik_n x} + B e^{-ik_n x}$$

with boundary conditions $\varphi_n(-\frac{a}{2}) = 0$ and $\varphi_n(+\frac{a}{2}) = 0$. These boundary conditions require

$$A = B \quad \text{with also} \quad k_n = \frac{n\pi}{a} \,, \quad n = 1, 3, 5 \dots$$

$$A = -B \quad \text{with also} \quad k_n = \frac{n\pi}{a} \,, \quad n = 2, 4, 6 \dots$$

which gives for the solutions $\varphi_n(x)$ (here already normalized)

$$\varphi_n(x) = \begin{cases} \sqrt{\frac{2}{a}} \cos(k_n x) \,, & k_n = \frac{n\pi}{a} \,, \; n = 1, 3, 5 \dots \\ \sqrt{\frac{2}{a}} \sin(k_n x) \,, & k_n = \frac{n\pi}{a} \,, \; n = 2, 4, 6, \dots \end{cases}$$

which are consistent with eigenvalues E_n according to

$$E_n = \frac{n^2 \hbar^2 \pi^2}{2ma^2}$$

2b) Dipole oscillations are described by 11/16

$$< \psi(t) \,|\hat{D}|\, \psi(t) > \; = \; < \hat{D}(t) >$$

For the system in a state $\alpha|\varphi_n\rangle + \beta|\varphi_n\rangle$ at some time $t=0$, this gives

$$< \hat{D}(t) > = \alpha^*\alpha < \varphi_n / \hat{D} / \varphi_n > + \beta^*\beta < \varphi_m / \hat{D} / \varphi_m >$$
$$+ e^{-\frac{i}{\hbar}(E_n - E_m)t} \beta^*\alpha < \varphi_m / \hat{D} / \varphi_n > +$$
$$e^{-\frac{i}{\hbar}(E_m - E_n)t} \alpha^*\beta < \varphi_n / \hat{D} / \varphi_m >$$

The only oscillating terms are governed (in amplitude) by $<\varphi_n|\hat{D}|\varphi_m>$ and $<\varphi_m|\hat{D}|\varphi_n> = <\varphi_n|\hat{D}|\varphi_m>^*$ wich are equal to inner products as

$$<\varphi_n|\hat{D}|\varphi_m> = \int_{-\frac{a}{2}}^{a/2} \varphi_n^*(x) \; \hat{D} \; \varphi_m(x) \; dx$$

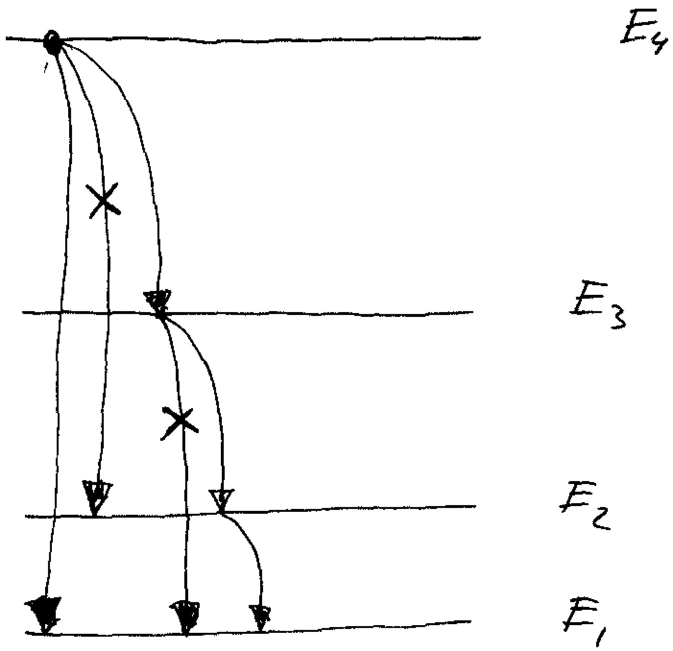
The solutions of 2a) all have even or odd parity, that is they are all symmetric or anti-symmetric in x.

The Dipole operator $\hat{D} = e\hat{X}$, is ex in x-representation, and is anti-symmetric in x.

Thus, integrals of the type $\int_{-a/2}^{a/2} \varphi_n^*(x)(ex)\,\varphi_m(x)\,dx$ yield zero, if $\varphi_n(x)$ and $\varphi_m(x)$ are both symmetric or both anti-symmetric.

Consequently, the system's dipole oscillations 12/16 have zero amplitude ($<\varphi_n|\hat{D}|\varphi_m> = 0$) if $\varphi_n(x)$ and $\varphi_m(x)$ are both symmetric or both anti-symmetric.

2c)



Transitions marked with X will not occur, since they are forbidden by parity (cannot lower energy, since a photon cannot be emitted)

For each transition, the energy of the emitted photon is

$$\hbar\omega_{nm} = E_n - E_m$$

The system can thus relax as follows:

(1) Directly from E_4 to E_1, by emitting a photon $\hbar\omega_{41} = E_4 - E_1$, final state is $\varphi_1(x)$

(2) From E_4 to E_2 is forbidden

(3) From E_4 to E_3 (and then emitting a photon $\hbar\omega_{43} = E_4 - E_3$), and then from E_3 to E_2 (directly to E_1 now forbidden) by emitting a photon $\hbar\omega_{32} = E_3 - E_2$, and then from E_2 to E_1, by emitting a photon $\hbar\omega_{21} = E_2 - E_1$. Final state is $\varphi_1(x)$

Problem 3

3a) A free-particle of mass m, in one dimension, has only kinetic energy $\Rightarrow$

$$\hat{H} = \frac{\hat{p}^2}{2m}$$, with $\hat{p}$ the momentum operator.

3b) $$\hat{U} = e^{-\frac{i}{\hbar}\hat{H}t} = e^{-\frac{i}{\hbar}\frac{\hat{p}^2}{2m}t}$$

3c) Plane-wave in positive x-direction $\Rightarrow$

$$\Psi_{PW}(x,t) = A\, e^{-i(kx - \omega t)}$$

probability amplitude

wave number $k = \frac{p}{\hbar} = \frac{2\pi}{\lambda_{\text{Broglie}}}$

$\hbar\omega = \frac{p^2}{2m}$ (kinetic energy)

3d)

$$\Psi(x,t) = \hat{U}\,\Psi(x, t=0)$$
$$= e^{-\frac{i}{\hbar}\frac{\hat{p}^2}{2m}t}\frac{1}{\sqrt{a}}\, e^{-|x|/a}$$

Using $p = \hbar k$, the operator $\hat{U}$ for a specific k-value is simply multiplication with a scalar number, so the state's time evolution is more easily evaluated in the k-representation.

Here it becomes a superposition of plane waves, each with time evolution as the state in 3c)

So, first Fourier transform $\psi(x,t=0)$

$$\psi(x,t=0) \overset{\mathcal{F}}{\longleftrightarrow} \bar{\psi}(k,t=0)$$

$$\bar{\psi}(k,t=0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x,t=0)\, e^{-ikx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \frac{1}{\sqrt{a}}\, e^{\frac{x}{a}} e^{-ikx}\, dx + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \frac{1}{\sqrt{a}}\, e^{-\frac{x}{a}} e^{-ikx}\, dx$$

$$= \frac{1}{\sqrt{2\pi a}} \left(\frac{1}{\frac{1}{a} - ik} \left[e^{(\frac{1}{a}-ik)x} \right]_{-\infty}^{\infty} + \frac{-1}{(\frac{1}{a}+ik)} \left[e^{-(\frac{1}{a}+ik)x} \right]_{0}^{\infty} \right)$$

$$= 2\sqrt{\frac{a}{2\pi}}\ \frac{1}{1+a^2k^2}$$

In k-representation $\hat{u} = e^{-\frac{i}{\hbar} \frac{\hbar^2 k^2}{2m} \cdot t}$

$$\Rightarrow \hat{u} = e^{-i\omega_k t} \quad \text{with } \omega_k = \frac{\hbar k^2}{2m}$$

So,

$$\psi(x,t) = \hat{u}\, \psi(x,t=0)$$

$$= e^{-\frac{i}{\hbar} \frac{\hat{p}^2}{2m} t} \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{\psi}(k)\, e^{ikx}\, dk}_{\text{inverse Fourier transform}}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega_k t}\ 2\sqrt{\frac{a}{2\pi}}\ \frac{1}{1+a^2k^2}\ e^{ikx}\, dk$$

$$= \frac{2\sqrt{a}}{2\pi} \int_{-\infty}^{\infty} \frac{1}{1+a^2k^2}\ e^{i(kx - \omega_k t)}\, dk$$

Superposition of plane waves, each with its time evolution, and each with probability amplitude $\frac{2\sqrt{a}}{2\pi} \cdot \frac{1}{1+a^2k^2}$.